Heat Transfer

Conduction - One Dimensional Heat Conduction Equation

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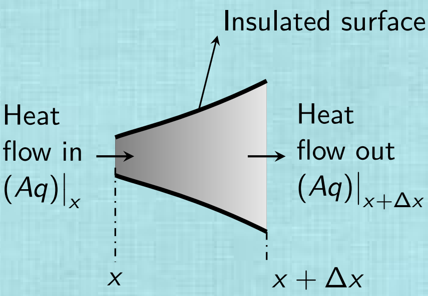
Objectives

- To derive the general one dimensional heat conduction equation.
- Reduce the above general equation to simple forms under various restricted conditions.
- To illustrate the variables of heat conduction—thermal conductivity, and, thermal diffusivity.
- To obtain the equations for heat conduction in terms of heat transfer resistance, for heat transfer through flat plate, hollow cylinder, and hollow sphere.

Outcome

- Understand the general form of heat conduction equation.
- Obtaining the heat conduction equation for a given set of conditions, from the general form.
- Deriving the equation of temperature profile for steady state heat conduction—for flat plate, cylinder, sphere.

One Dimensional Heat Conduction Equation



Let us consider a volume element of thickness Δx and having an area A normal to the coordinate axis x, as shown in the figure. The energy balance equation for this volume element is given by:

$$\underbrace{\begin{pmatrix} \text{Net rate of} \\ \text{heat gain by} \\ \text{conduction} \end{pmatrix}}_{\text{I}} + \underbrace{\begin{pmatrix} \text{rate of} \\ \text{energy} \\ \text{generation} \end{pmatrix}}_{\text{II}} = \underbrace{\begin{pmatrix} \text{rate of} \\ \text{increase of} \\ \text{internal energy} \end{pmatrix}}_{\text{III}} \quad (1)$$

One Dimensional Heat Conduction Equation (contd..)

The net rate of heat gain by the element by conduction is given by

$$\mathrm{I} = (Aq)\big|_x - (Aq)\big|_{x+\Delta x}$$

The rate of energy generation in the element having a volume $A\Delta x$ is given by

$$\mathrm{II} = A\Delta x \dot{g}$$

where $\dot{g} = \dot{g}(x,t)$ is the rate of energy generation per unit volume. The rate of increase of internal energy of the internal energy of the volume element resulting from the change of temperature with unit time is given by

$$\mathrm{III} = A\Delta x \, \rho C_P \frac{\partial T(x,t)}{\partial t}$$

(Note: Internal energy change is $mC_V\Delta T$; but for liquids and solids, $C_P \approx C_V$).

One Dimensional Heat Conduction Equation (contd..)

Substituting for the quantities in Eqn.(1), and rearranging, we get

$$-\frac{1}{A}\frac{\left((Aq)\big|_{x+\Delta x} - (Aq)\big|_{x}\right)}{\Delta x} + \dot{g} = \rho C_P \frac{\partial T(x,t)}{\partial t}$$

As $\Delta x \to 0$,

$$\frac{\left((Aq)\big|_{x+\Delta x} - (Aq)\big|_{x}\right)}{\Delta x} = \frac{\partial}{\partial x}(Aq) \qquad \left(\begin{array}{c}\text{from the definition}\\ \text{of derivative}\end{array}\right)$$

And, $q = -k\dfrac{\partial T}{\partial x}$. Therefore, the above equation becomes

$$\boxed{\frac{1}{A}\frac{\partial}{\partial x}\left(Ak\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C_P \frac{\partial T(x,t)}{\partial t}} \qquad (2)$$

This is the general equation for one dimensional heat conduction.

Rectangular Coordinates:
Here, the area A does not vary with x. Hence, Eqn.(2) becomes,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C_P \frac{\partial T(x,t)}{\partial t}$$

Cylindrical Coordinates:
Here, $x = r$. Area, $A = 2\pi rL$. i.e., $A \propto r$. Hence, Eqn.(2) becomes,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C_P \frac{\partial T(r,t)}{\partial t}$$

Spherical Coordinates:
Here too, $x = r$. Area, $A = 4\pi r^2$. i.e., $A \propto r^2$. Hence, Eqn.(2) becomes,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C_P \frac{\partial T(r,t)}{\partial t}$$

The above equations (rectangular / cylindrical / spherical coordinates) can be written in a compact form, as below:

$$\boxed{\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C_P \frac{\partial T}{\partial t}} \qquad (3)$$

where

$$n = \begin{cases} 0 & \text{for rectangular coordinates} \\ 1 & \text{for cylindrical coordinates} \\ 2 & \text{for spherical coordinates} \end{cases}$$

And, in the rectangular coordinates system, it is customary to use the variable x in place of r.

For constant thermal conductivity k, Eqn.(3) reduces to,

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right)+\frac{1}{k}\dot{g}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where

$$\alpha=\frac{k}{\rho C_P}=\text{thermal diffusivity of material, m}^2/\text{s}$$

For steady state heat conduction with energy sources within the medium, Eqn.(3) reduces to,

$$\frac{1}{r^n}\frac{d}{dr}\left(r^n k\frac{dT}{dr}\right)+\dot{g}=0$$

and for the case of conduction with constant k,

$$\frac{1}{r^n}\frac{d}{dr}\left(r^n\frac{dT}{dr}\right)+\frac{1}{k}\dot{g}=0$$

For steady state conduction, with no energy sources, and for constant k,

$$\frac{d}{dr}\left(r^n \frac{dT}{\partial r}\right) = 0$$

Thermal Diffusvity (α)

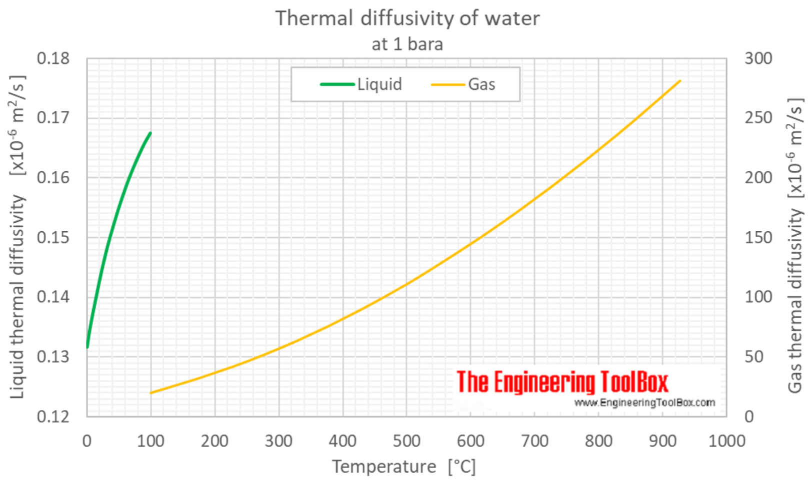
$$\boxed{\alpha = \frac{k}{\rho C_P}}$$

- Thermal diffusivity is a measure of the transient thermal response of a material to a change in temperature.
- The larger the value of α, the faster will the heat diffuse through the material and its temperature will change with time.
- This will result either due to a high value of conductivity k or a low value of ρ, C_P.
- Thermal diffusivity is a convenient collection of physical properties for transient solutions of the heat equation.
- Recollect about 'kinematic viscosity' ($\nu = \mu/\rho$), which is also called as 'momentum diffusivity'; and D_{AB}.

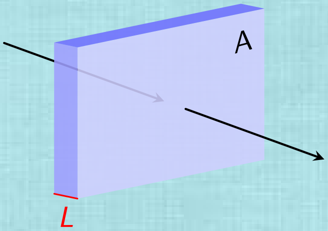
Thermal Diffusivity (contd..)

Material	Thermal Diffusivity (cm^2/s) at 300 K
Copper	1.15
Aluminum	0.97
Stainless Steel (304)	0.042
Silicon Dioxide (Polycrystalline)	0.0083
Water	0.0014
Polyvinyl Chloride (PVC)	0.0008
Alcohol	0.0007
Air	0.19

- Metals and gases have relatively high value of thermal diffusivity and their response to temperature changes is quite rapid.
- The non metallic solids and liquids respond slowly to temperature changes because of their relatively small value of thermal diffusivity.


Thermal diffusivity of water
at 1 bara
Liquid
Gas
Liquid thermal diffusivity [x10-6 m2/s]
Gas thermal diffusivity [x10-6 m2/s]
0.18
0.17
0.16
0.15
0.14
0.13
0.12
300
250
200
150
100
50
0
0 100 200 300 400 500 600 700 800 900 1000
Temperature [°C]
The Engineering ToolBox
www.EngineeringToolBox.com

One Dimensional Steady State Heat Conduction through Flat Plate



Consider the system shown above. The top, bottom, front and back of the cube are insulated, so that heat can be conducted through the cube only in the x direction. In this special case, heat flow is one dimensional. If sides were not insulated, heat flow could be two or three dimensional.

Boundary conditions:

$$T = T_1 \qquad \text{at} \quad x = 0$$
$$T = T_2 \qquad \text{at} \quad x = L$$

Conduction through Flat Plate (contd..)

$$\frac{d}{dx}\left(\frac{dT}{\partial x}\right) = 0$$

Intergrating , we get

$$\frac{dT}{dx} = C_1 \qquad (1)$$

Intergrating further, we get

$$T = C_1 x + C_2 \qquad (2)$$

Using the boundary condition at $x = 0$ gives

$$C_2 = T_1$$

And, from the boundary condition at $x = L$ gives,

$$T_2 = C_1 L + T_1 \qquad \Longrightarrow \qquad C_1 = \frac{T_2 - T_1}{L}$$

Substituting for C_1 in Eqn.(1), we get

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

From the definition of heat flux (as given by Fourier's law)

$$q = -k\frac{dT}{dx}$$

Therefore,

$$q = -k\frac{T_2 - T_1}{L} = k\frac{T_1 - T_2}{L}$$

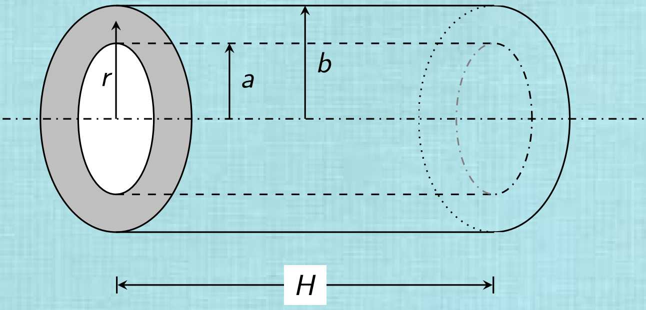
$$Q = qA = kA\frac{T_1 - T_2}{L} = \frac{T_1 - T_2}{R}$$

where

$$\boxed{R = \frac{L}{kA}}$$

R is called the thermal resistance of the flat plate for heat flow through an area of A across a temperature difference of $T_1 - T_2$.

One Dimensional Steady State Heat Conduction through Cylindrical Surface



$$
\begin{aligned}
T &= T_1 \qquad \text{at} \quad r = a \\
T &= T_2 \qquad \text{at} \quad r = b
\end{aligned}
$$

Conduction through Cylindrical Surface (contd..)

$$\frac{d}{dr}\left(r\frac{dT}{\partial r}\right) = 0$$

Intergrating , we get

$$r\frac{dT}{dr} = C_1$$

i.e.,

$$\frac{dT}{dr} = \frac{C_1}{r} \tag{1}$$

Intergrating further, we get

$$T = C_1 \ln r + C_2 \tag{2}$$

Using the boundary condition at $r = a, b$ gives

$$T_1 = C_1 \ln a + C_2 \tag{3}$$

$$T_2 = C_1 \ln b + C_2 \tag{4}$$

$$\text{Eqn.}(4) - \text{Eqn.}(3) \implies \qquad T_2 - T_1 = C_1(\ln b - \ln a)$$

i.e.,

$$C_1 = -\frac{T_1 - T_2}{\ln(b/a)} \tag{5}$$

Using Eqn.(5) in Eqn.(1), we get

$$\frac{dT}{dr} = \frac{C_1}{r} = -\frac{1}{r}\frac{T_1 - T_2}{\ln(b/a)}$$

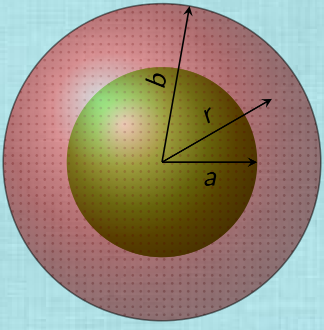
From the definition of heat flux, $q = -k\dfrac{dT}{dr}$ And, from $Q = qA = q(2\pi rH)$, we get

$$Q = 2\pi kH\frac{T_1 - T_2}{\ln(b/a)} = \frac{T_1 - T_2}{R}$$

where

$$\boxed{R = \frac{\ln(b/a)}{2\pi kH}}$$

One Dimensional Steady State Heat Conduction through Spherical Surface



$$T = T_1 \qquad \text{at} \quad r = a$$
$$T = T_2 \qquad \text{at} \quad r = b$$

$$\frac{d}{dr}\left(r^2 \frac{dT}{\partial r}\right) = 0$$

Intergrating , we get

$$r^2 \frac{dT}{dr} = C_1$$

i.e.,

$$\frac{dT}{dr} = \frac{C_1}{r^2} \tag{1}$$

Intergrating further, we get

$$T = -\frac{C_1}{r} + C_2 \tag{2}$$

Using the boundary condition at $r = a, b$ gives

$$T_1 = -\frac{C_1}{a} + C_2 \tag{3}$$

$$T_2 = -\frac{C_1}{b} + C_2 \tag{4}$$

Conduction through Spherical Surface (contd..)

Eqn.(3) − Eqn.(4) $\Longrightarrow$

$$T_1 - T_2 = \frac{C_1}{b} - \frac{C_1}{a} = \frac{aC_1 - bC_1}{ab} = -\frac{b-a}{ab} C_1$$

i.e.,

$$C_1 = -\frac{ab}{(b-a)}(T_1 - T_2)$$

From the definition of heat flux, $q = -k\frac{dT}{dr}$, and from Eqn.(1), we get

$$q = \frac{k}{r^2}\frac{ab}{(b-a)}(T_1 - T_2)$$

$$Q = qA = q(4\pi r^2) \quad \Longrightarrow \quad Q = \frac{4\pi kab}{(b-a)}(T_1 - T_2) = \frac{T_1 - T_2}{R}$$

where

$$\boxed{R = \frac{b-a}{4\pi kab}}$$

Heat Transfer Resistance - Generalization

Rate of heat conduction through any sort of surface, from surface 1 to surface 2 can be given by

$$Q = \frac{T_1 - T_2}{R}$$

with

$$R = \frac{\Delta x}{k\, A_m}$$

where, Δx = thickness of surface through which heat is getting transferred; and, A_m = mean heat transfer area.

$$A_m = \begin{cases} A_{am} & = \text{arithmetic mean, for rectangular coordinates} \\ A_{lm} & = \text{logarithmic mean, for cylindrical coordinates} \\ A_{gm} & = \text{geometric mean, for spherical coordinates} \end{cases}$$

Heat Transfer Resistance (contd..)

For flat surface,

$$A_m = A_{am} = \frac{A_1 + A_2}{2} = \frac{A + A}{2} = A$$

For cylindrical surface,

$$A_m = A_{lm} = \frac{A_1 - A_2}{\ln \frac{A_1}{A_2}} = \frac{2\pi aH - 2\pi bH}{\ln \frac{2\pi aH}{2\pi bH}} = \frac{2\pi H(b - a)}{\ln(b/a)} = \frac{2\pi H \Delta x}{\ln(b/a)}$$

For spherical surface,

$$A_m = A_{gm} = \sqrt{A_1 A_2} = \sqrt{(4\pi a^2)(4\pi b^2)} = 4\pi ab$$

During steady one-dimensional heat conduction in a spherical (or cylindrical) container, the rate of heat transfer (Q) remains constant, but the heat flux (q) decreases with increasing radius.

Temperature profile for one dimensional heat flow ($T_1 > T_2$ and $b > a$)

Flat plate

$$T(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

Cylinder:

$$\frac{T(r) - T_1}{T_2 - T_1} = \frac{\ln(r/a)}{\ln(b/a)}$$

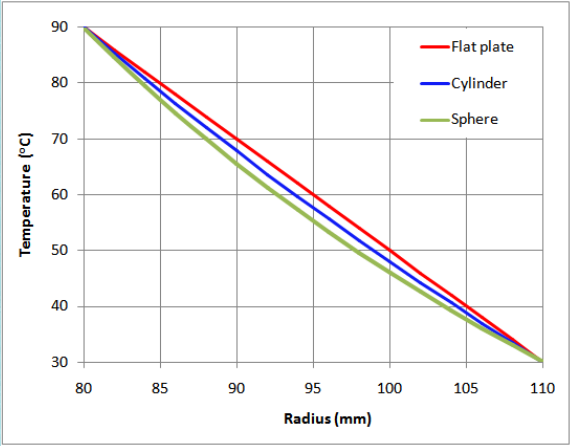
Sphere:

$$T(r) = \frac{a}{r} \cdot \frac{b-r}{b-a} \cdot T_1 + \frac{b}{r} \cdot \frac{r-a}{b-a} \cdot T_2$$

Temperature Profile for One Dimensional Steady State Heat Conduction

- For flow across flat surfaces: T vs. x is linear; $\dfrac{dT}{dx} =$ constant.
- For flow across cylindrical surfaces: T vs. $\ln r$ is linear; $\dfrac{dT}{dr} \propto \dfrac{1}{r}$
- For flow across spherical surfaces: T vs. $1/r$ is linear; $\dfrac{dT}{dr} \propto \dfrac{1}{r^2}$

Temperature Profile for One Dimensional Steady State Heat Conduction (contd..)



The one-dimensional unsteady heat conduction equation is

$$\rho C_P \frac{\partial T}{\partial t} = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n k \frac{\partial T}{\partial r} \right)$$

where T - temperature, t - time, r - radial position, k - thermal conductivity, ρ - density, and C_P - specific heat.

For the cylindrical coordinate system, the value of n in the above equation is (G-2017-12)

(a) 0 (b) 1 (c) 2 (d) 3

(b) ✓ *Explanation:* $n = 1$ for cylindrical coordinate system.

$n = 2$ for spherical; and 0 for rectangular coordinate systems. And, r will be replaced with x for rectangular system. □

The heat flux (from outside to inside) across an insulating wall with thermal conductivity $k = 0.04$ W/m.K and thickness 0.16 m is 10 W/m^2. The temperature of the inside wall is $-5°$C. The outside wall temperature is (G-2001-2.11)

(a) 25°C (b) 30°C (c) 35°C (d) 40°C

(c) ✓ *Explanation:* $q = k\dfrac{\Delta T}{\Delta L} \implies \Delta T = \dfrac{10 \times 0.16}{0.04} = 40°\text{C}.$

Therefore outside temperature $= -5 + 40 = 35°\text{C}.$ □

A stagnant liquid film of 0.4 mm thickness is held between two parallel plates. The top plate is maintained at 40°C and the bottom plate is maintained at 30°C. If the thermal conductivity of the liquid is 0.14 W/(m.K), then the steady state heat flux (in W/m^2) assuming one dimensional heat transfer is (G-2006-10)

(a) 3.5 (b) 350 (c) 3500 (d) 7000

(c) ✓ *Explanation:* Heat flux (q) is given by

$$q = k\frac{\Delta T}{\Delta x} = 0.14 \times \frac{(40 - 30)}{0.4 \times 10^{-3}} = 3500 \text{ W/m}^2 \qquad \square$$

A circular tube of outer diameter 5 cm and inner diameter 4 cm is used to convey hot fluid. The inner surface of the wall of the tube is at a temperature of 80°C, while the outer surface of the wall of the tube is at 25°C. What is the rate of heat transport across the tube wall per meter length of the tube of steady state, if the thermal conductivity of the tube wall is 10 W/(m.K) ? (G-2005-58)

(a) 13823 W/m (b) 15487 W/m (c) 17279 W/m (d) 27646 W/m

(b) ✓ *Explanation:* Rate of heat loss (Q) is given by

$$Q = kA\frac{\Delta T}{t}$$

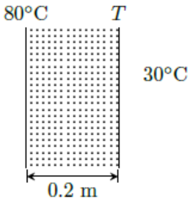
For the cylindrical tube, $A = A_{\mathrm{lm}} = \dfrac{2\pi L(r_o - r_i)}{\ln(r_o/r_i)}$. Therefore, for unit length of pipe,

$$Q = \frac{k2\pi(r_o - r_i)}{\ln(r_o/r_i)}\frac{\Delta T}{r_o - r_i} = \frac{10 \times 2\pi}{\ln(2.5/2)} \times (80 - 25) = 15487 \text{ W/m} \qquad \square$$

The left face of a one dimensional slab of thickness 0.2 m is maintained at 80°C and the right face is exposed to air at 30°C. The thermal conductivity of the slab is 1.2 W/(m.K) and the heat transfer coefficient from the right face is 10 W/(m^2.K). At steady state, the temperature of the right face in °C is (G-2004-58)

(a) 77.2 (b) 71.2 (c) 63.8 (d) 48.7

(d) ✓ *Explanation:*



At steady state, rate of heat transfer by conduction through the slap is equal to the rate of heat transfer by convection in the air. i.e.,

$$k\,\frac{\Delta T_1}{\Delta x} = h\,\Delta T_2$$

Substituting the known data, we get

$$\frac{1.2 \times (80 - T)}{0.2} = 10 \times (T - 30)$$

Solving, we get $T = 48.7$°C.

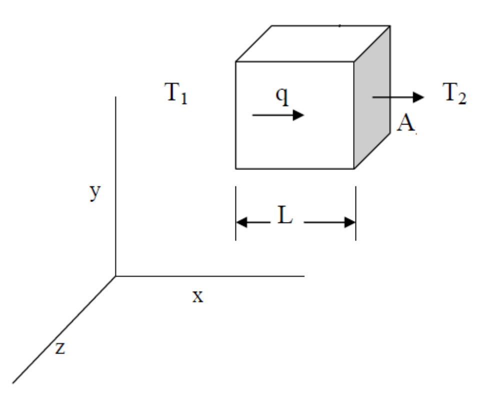
A layer of insulation is applied on a spherical metallic tank containing a cryogenic liquid. The rate of heat leakage into the tank

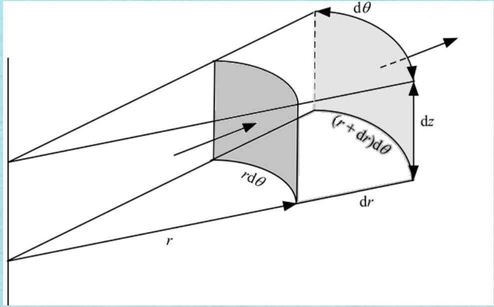
(a) always decreases with increase in insulation thickness

(b) increases with insulation thickness and remains constant beyond a critical thickness

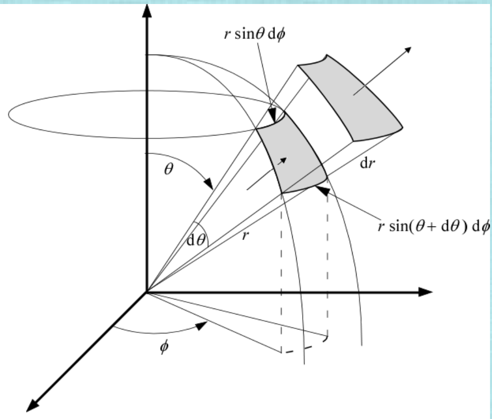
(c) may increase up to a thickness and then decreases, depending upon the thermal conductivity of the insulating material

(d) may increase up to a thickness and then decrease depending upon the thermal conductivity of the metal


T1
q
T2
A
y
L
x
z



dθ
dz
$(r + \mathrm{d}r)\mathrm{d}\theta$
$r\mathrm{d}\theta$
dr
r



$r \sin\theta \, d\phi$
θ
$d\theta$
r
dr
$r \sin(\theta + d\theta) \, d\phi$
ϕ

Three Dimensional Heat Flow (contd..))

Rectangular (Cartesian) coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

Boundary and Initial Conditions

We have set up a differential equation, with T as the dependent variable. The solution will give us $T(x, y, z)$. Solution depends on boundary conditions (BC) and initial conditions (IC).

How many BC's and IC's? Heat equation is second order in spatial coordinate. Hence, 2 BC's are needed for each coordinate.

- 1D problem: 2 BC in x-direction.
- 2D problem: 2 BC in x-direction, 2 in y-direction.
- 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.

Heat equation is first order in time. Hence one IC needed.

Questions for Practice

1. Starting from general one dimensional heat conduction equation, obtain the following expressions, for steady state heat transfer through flat plate.
   At $x = 0$, $T = T_1$; and, at $x = L$, $T = T_2$. With $T_1 > T_2$.
   $$\boxed{\frac{T(x) - T_1}{T_2 - T_1} = \frac{x}{L}}$$
2. Starting from general one dimensional heat conduction equation, obtain the following expressions, for steady state heat transfer through cylindrical shell.
   At $r = r_1$, $T = T_1$; and, at $r = r_2$, $T = T_2$. With $T_1 > T_2$, and $r_2 > r_1$. $Q = qA$.
   (a) $T(r) = \frac{q_1 r_1}{k} \ln\left(\frac{r_2}{r}\right) + T_2$
   (b) $T(r) = \frac{q_2 r_2}{k} \ln\left(\frac{r_1}{r}\right) + T_1$
   (c) $\boxed{\frac{T(r) - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}}$

3. Obtain the following for spherical shell, with the conditions as above:
   (a) $T(r) = \dfrac{q_1 r_1^2}{k} \ln\left(\dfrac{1}{r} - \dfrac{1}{r_2}\right) + T_2$
   (b) $T(r) = \dfrac{q_2 r_2^2}{k} \ln\left(\dfrac{1}{r} - \dfrac{1}{r_1}\right) + T_1$
   (c) $\boxed{\dfrac{T(r) - T_1}{T_2 - T_1} = \dfrac{\left(\dfrac{1}{r} - \dfrac{1}{r_1}\right)}{\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)}}$