CH6605 Process Instrumentation, Dynamics and Control

Routh-Hurwitz Stability Criterion

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Introduction

- A dynamic system is said to be stable if for every bounded input it produces a bounded output, regardless of its initial state. Bounded is an input that always remains between an upper and a lower limit (e.g.: sinusoidal, step, but not the ramp).
- If the transfer function of a dynamic system has even one pole with positive real root, the system is unstable.

Routh-Hurwitz Criterion for Stability

The criterion of stability for the closed-loop systems does not require calculations of the actual values of the roots of the characteristic polynomial. It only requires that we know if any root is to the right of the imaginary axis.

For the characteristic equation (with a_0 as positive)

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

First test: If any of the coefficients of the characteristic equation $a_1, a_2, \ldots, a_{n-1}, a_n$ is negative, then there is at least one root of the characteristic equation which has a positive real part, and the corresponding system is unstable. No further analysis is needed.

Routh-Hurwitz Criterion for Stability (contd..)

Second test: If all coefficients $a_1, a_2, \ldots, a_{n-1}, a_n$ are positive, then the following analysis by using Routh array is to be made. The elements of Routh array are written as

Row 1	a_0	a_2	a_4	a_6	$\cdots$
2	a_1	a_3	a_5	a_7	$\cdots$
3	A_1	A_2	A_3	$\cdot$	$\cdots$
4	B_1	B_2	B_3	$\cdot$	$\cdots$
5	C_1	C_2	C_3	$\cdot$	$\cdots$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$
$n+1$	V_1	V_2	$\cdot$	$\cdot$	$\cdots$

where

$$A_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad A_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad A_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad \cdots$$

$$B_1 = \frac{A_1 a_3 - a_1 A_2}{A_1} \qquad B_2 = \frac{A_1 a_5 - a_1 A_3}{A_1} \qquad \cdots$$

$$C_1 = \frac{B_1 A_2 - A_1 B_2}{B_1} \qquad C_2 = \frac{B_1 A_3 - A_1 B_3}{B_1} \qquad \cdots$$

etc.

Routh-Hurwitz Criterion for Stability (contd..)

Examine the elements of the first column of the Routh array:

$$a_0 \quad a_1 \quad A_1 \quad B_1 \quad C_1 \quad \dots \quad V_1$$

(a) If any of these elements is negative, we have atleast one root to the right of the imaginary axis and the system is unstable.

(b) The number of sign changes in the elements of the first column is equal to the number of roots to the the right of the imaginary axis.

Therefore, a system is stable if all the elements in the first column of the Routh array are positive.

Solved Problems

Example 1: Checking for Stability

The characteristic equation of a closed loop control system is

$$s^4 + 4s^3 + 6s^2 + 2s + 3 = 0$$

Check whether the system is stable or not. (G-1994-28)

Solution:

Characteristic equation:

$$s^4 + 4s^3 + 6s^2 + 2s + 3 = 0$$

Routh array:

$$\begin{array}{lll}
\text{Row } 1 & 1 & 6 & 3 \\
2 & 4 & 2 & \\
3 & \dfrac{4 \times 6 - 1 \times 2}{4} = 5.5 & \dfrac{4 \times 3 - 1 \times 0}{4} = 3 & \\
4 & \dfrac{5.5 \times 2 - 4 \times 3}{5.5} = -\dfrac{1}{5.5} & &
\end{array}$$

Since the first column of Routh array is having negative element (i.e., $-1/5.5$), the given system is unstable. □

Example 2: Maximum Controller Gain for Stability

The characteristic equation of a closed loop system using a proportional controller with gain K_c is

$$12s^3 + 19s^2 + 8s + 1 + K_c = 0$$

At the onset of instability, the value of K_c is (G-2009-42)

(a) $35/3$ (b) 10 (c) $25/3$ (d) $20/3$

Solution:

The Routh array of the given characteristic equations is given below:

Row 1	12	8
2	19	$1 + K_c$
3	$\dfrac{19 \times 8 - 12 \times (1 + K_c)}{19}$	
4	$1 + K_c$	

For stability all the elements in the first column of Routh array should be positive. Therefore,

$$\frac{19 \times 8 - 12 \times (1 + K_c)}{19} > 0$$

$$\text{i.e.,} \qquad 152 > 12 + 12K_c$$

$$\implies \qquad K_c < 35/3$$

At values of $K_c > 35/3$, the closed loop system is unstable. □

Example 3: Number of Negative Roots

Given the characteristic equation below, select the number of roots which will be located to the right of the imaginary axis (G-2005-47)

$$s^4 + 5s^3 - s^2 - 17s + 12 = 0$$

(a) One (b) Two (c) Three (d) Zero

Solution:

The characteristic equation is:

$$s^4 + 5s^3 - s^2 - 17s + 12 = 0$$

From the first test of Routh analysis, it is found that the given system is unstable. To find the number of roots to the right of the imaginary axis, the second test of Routh analysis is to be done.

Elements of Routh array are calculated as follows:

Row 1	1	-1	12
2	5	-17	0
3	$\dfrac{5\times(-1)-1\times(-17)}{5}=2.4$	$\dfrac{5\times 12-1\times 0}{5}=12$	0
4	$\dfrac{2.4\times(-17)-5\times 12}{2.4}=-42$	0	0
5	$\dfrac{(-42)\times(12)-2.4\times 0}{(-42)}=12$	0	0

The elements of the first column of Routh array are: $[1 \quad 5 \quad 2.4 \quad -42 \quad 12]$. There are two changes here: one from 2.4 to -42 (i.e., positive to negative), another from -42 to 12 (i.e., negative to positive). Hence, there are two roots to the right of the imaginary axis. **(b)** ✓

$$s^4 + 5s^3 - s^2 - 17s + 12 = 0$$

```
» p = [1 5 -1 -17 12];
» roots(p)
ans = -4 -3 1 1
```

The standard procedure fails if we encounter any of the following situations in the formulation of the array.

1. A row of all zeros appears.
2. First element of a row, appearing in first column of the array is zero, but the entire row is not all zeros.

Zero in the First Column of Routh Array

When a zero occurs in the first column of Routh array, create a Routh table using the polynomial that has the reciprocal roots of the original polynomial.
e.g.: For the polynomial, $s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3 = 0$, we will get the Routh table with first column to be zero.
By replacing s with $1/s$, we get

$$\frac{1}{s^5} + \frac{2}{s^4} + \frac{3}{s^3} + \frac{6}{s^2} + \frac{5}{s} + 3 = 1 + 2s + 3s^2 + 6s^3 + 5s^4 + 3s^5$$

```
» p = [1 2 3 6 5 3];
p =
1 2 3 6 5 3
» roots(p)
ans =
0.3429 + 1.5083i
0.3429 - 1.5083i
-1.6681 + 0.0000i
-0.5088 + 0.7020i
-0.5088 - 0.7020i
»
```