CH6605 Process Instrumentation, Dynamics and Control

Pole Zero Analysis

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Pole Zero Analysis

- The dynamic output $y(t)$ of the system for changes in the input $x(t)$ can be written in a polynomial form as:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{(1)} + a_o y = b_m x^{(m)} + b_{m-1} x^{(m-1)} + \cdots + b_1 x^{(1)} + b_o x$$

where $y^{(n)}$ is the n^{th} derivative of y. With $n \geq m$, and with zero initial conditions, i.e., $y^{(n)} = y^{(n-1)} = y = 0$ at $t = 0$, we can write the above differential equation using Laplace transform as below:

$$\frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_o}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_o} = G(s) = \frac{Q(s)}{P(s)}$$

This ratio $\frac{Y(s)}{X(s)} = G(s)$ is called as the transfer function of the system.

The above equation (given in previous slide) in pole-zero form is given as

$$G(s) = \frac{Q(s)}{P(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where p are poles and z are zeros of the transfer function. The zeros affect only the coefficients of the solution $y(t)$, but not the time dependent functions. Therefore in qualitative discussions, the focus is only on the poles.

Zeros affect the overall shape of the response.

Pole Zero Analysis

- 'Poles' are values of s that make the denominator of transfer function zero. Similarly, 'zeros' are values of s that make the numerator of transfer function zero.
- At a'pole' the transfer function of the system becomes infinite. At a 'zero' the transfer function of the system becomes 0.
- The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics. Together with the gain constant K they completely characterize the differential equation, and provide a complete description of the system.

Example

$$G(s) = \frac{K(s+1)(s+2)}{(s+1)(s+3)(s+4)}$$

zeros at: $-1, -2$

poles at: $-1, -3, -4$

Example: System Response

$$G(s) = \frac{6}{(s+1)(s+2)}, \qquad R(s) = \frac{1}{s+3}$$

$$C(s) = G(s)R(s) = \frac{6}{(s+1)(s+2)} \times \frac{1}{s+3}$$

$$= \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}$$

$$c(t) = 3\left\{\underbrace{e^{-t} - 2e^{-2t}}_{\substack{natural \\ response}} + \underbrace{e^{-3t}}_{\substack{forced \\ response}}\right\}, t \geq 0$$

For a physically realizable system, the order of the numerator of transfer function must be less than or equal to that of the denominator. e.g.:

$$G_1(s) = \frac{(s+1)(s+2)(s+3)}{(s+4)(s+5)} \qquad \text{(Physically unrealizable)}$$

$$G_1(s) = \frac{(s+1)(s+2)}{(s+4)(s+5)} \qquad \text{(Physically realizable)}$$

Nature of terms in the solution $x(t)$ based on roots in the denominator of $X(s)$

Roots in denominator of $X(s)$	Terms in $x(t)$ for $t > 0$
s_1	$C_1 e^{-a_1 t}$
s_2, s_2^*	$e^{-a_2 t}(C_1 \cos b_2 t + C_2 \sin b_2 t)$
s_3, s_3^*	$C_1 \cos b_3 t + C_2 \sin b_3 t$
s_4, s_4^*	$e^{a_4 t}(C_1 \cos b_4 t + C_2 \sin b_4 t)$
s_5	$C_1 e^{a_5 t}$
s_6	C_1

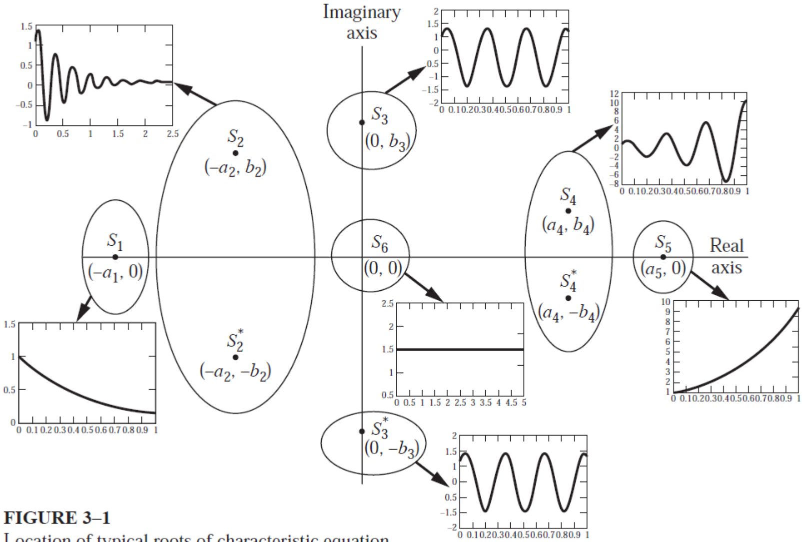


FIGURE 3–1
Location of typical roots of characteristic equation.

Example 2: ***Pole Zero Representation of a System***

A system has a pair of complex conjugate poles $p_1, p_2 = -1 \pm 2j$*, a single real zero* $z_1 = -2$*, and a gain factor* $K = 3$*. Find the differential equation representing the system. Represent the poles and zeros of the system in the* s *plane.*

Solution:
The transfer function of the system is :

$$\begin{aligned} G(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} = 3\frac{s-(-2)}{\big(s-(-1+2j)\big)\big(s-(-1-2j)\big)} \\ &= 3\frac{s+2}{s^2-s(-1-2j)-s(-1+2j)+(-1+2j)(-1-2j)} \\ &= 3\frac{s+2}{s^2+2s+\big((-1)^2-(2j)^2\big)} = 3\frac{s+2}{s^2+2s+5} \qquad (\text{as } j^2=-1) \end{aligned}$$

i.e.,

$$G(s) = \frac{Y(s)}{X(s)} = \frac{3s+6}{s^2+2s+5} \qquad \Longrightarrow \qquad (s^2+2s+5)Y(s) = (3s+6)X(s)$$

We know, that $\mathcal{L}^{-1}\left\{s^n Y(s)\right\} = \dfrac{d^n y}{dt}$. Hence,

$$(s^2+2s+5)Y(s) = \frac{d^2y}{dt} + 2\frac{dy}{dt} + 5y$$

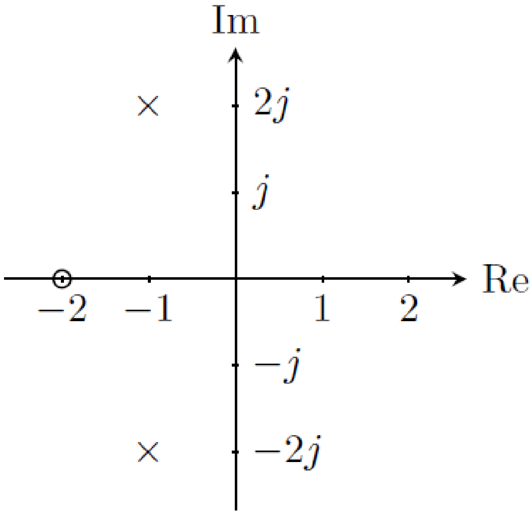
and,

$$(3s+6)X(s) = 3\frac{dx}{dt} + 6x$$

Hence, the differential equation representing the system is:

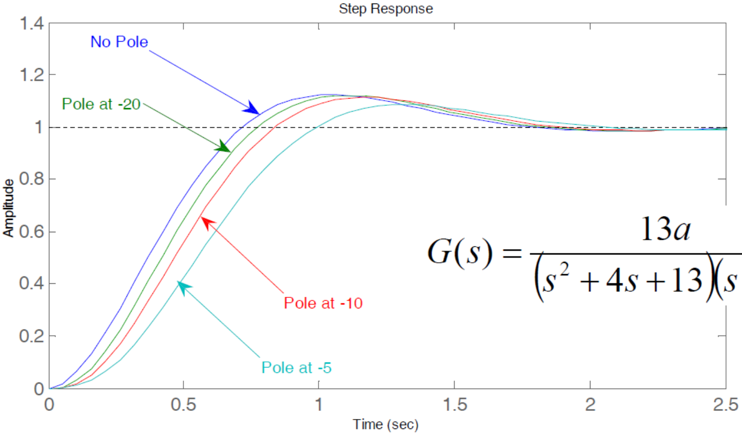
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{dx}{dt} + 6x$$

The poles (marked as $\times$) and zero (marked as $\circ$) are represented in the s plane as follows:



Effect of Pole

Time Response: 3rd Order System



$$G(s) = \frac{13a}{(s^2 + 4s + 13)(s + a)}$$

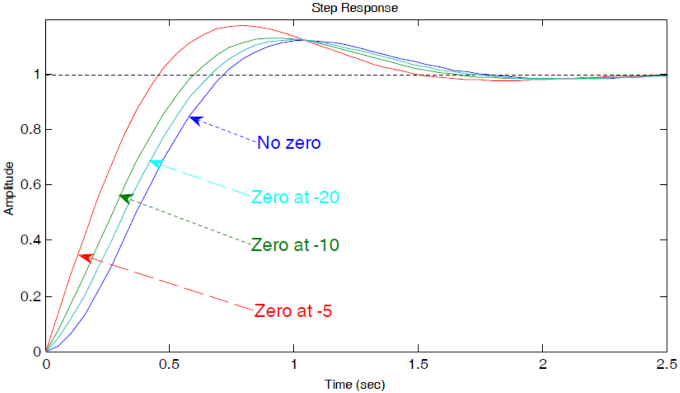
Effect of Pole (contd..)

- Slower response. Decreases overshoot.
- Reduced effect for pole farther in LHP.

Effect of Zero

Time Response: 2nd Order System with Zero

$$G(s) = \frac{(13/a)(s+a)}{s^2+4s+13}$$



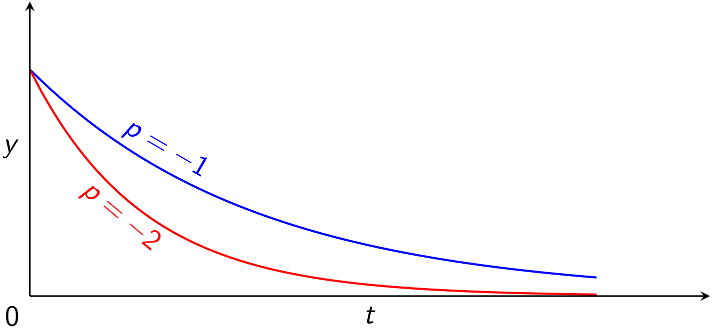
Effect of zero (contd..)

- Faster response
- Increased overshoot
- Reduced effect for zero farther in LHP

Negative Real Poles

$$Y(s) = \frac{1}{s+a} = \frac{1}{s-p} \qquad \text{where } p = -a$$

$$\boxed{Y = e^{-at}}$$

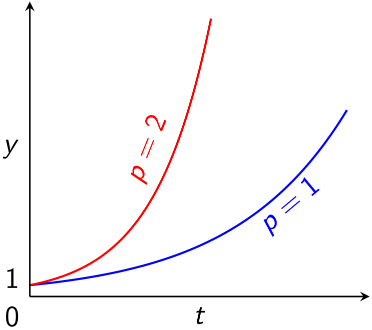


- Response decreases with time.
- Higher the negative value, faster is the decline.

Positive Real Poles

$$Y(s) = \frac{1}{s-a} = \frac{1}{s-p} \qquad \text{where } p = a$$

$$\boxed{Y = e^{at}}$$



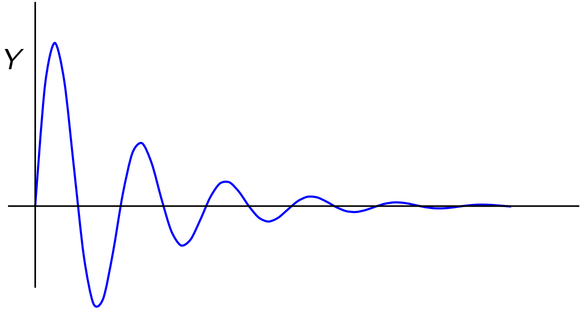
- Response grows with time.
- Higher the positive value, faster is the growth.

Complex Poles with Negative Real Parts

$$Y(s) = \frac{1}{s - p} \qquad \text{where } p = \alpha \pm j\beta$$

$$Y = e^{\alpha t} \sin(\beta t + \phi)$$

α is negative



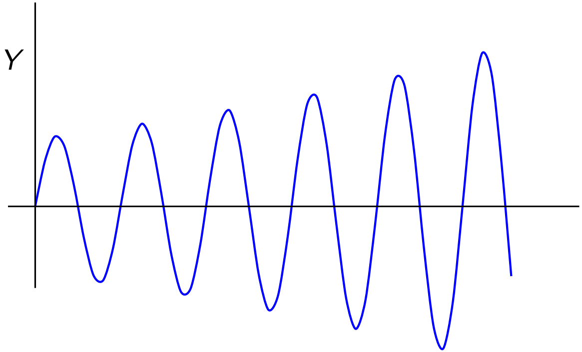
- Response declines with time in an oscillating manner with ever-decreasing amplitude.

Complex Poles with Positive Real Parts

$$Y(s) = \frac{1}{s - p} \qquad \text{where } p = \alpha \pm j\beta$$

$$Y = e^{\alpha t} \sin(\beta t + \phi)$$

α is positive



- Response increases with time in an oscillating manner with ever increasing amplitude