UCH1603 Process Dynamics and Control

First Order Systems in Series

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February 10, 2021

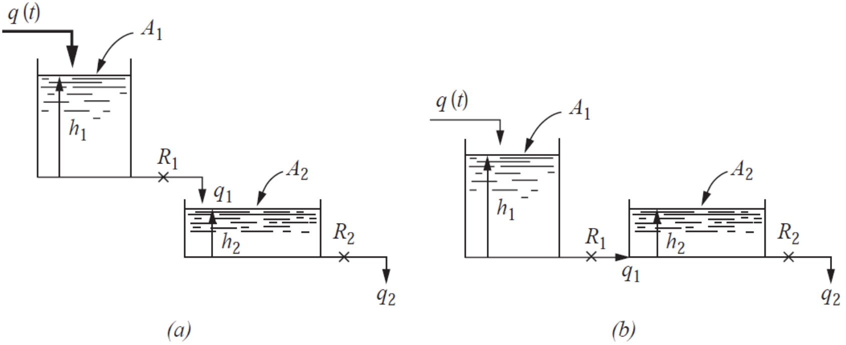

FIGURE 6–1
Two-tank liquid-level system: *(a)* Noninteracting; *(b)* interacting.

(a) Non-interacting Systems in Series

Tank-1:

Material balance:

$$q - q_1 = A_1 \frac{dh_1}{dt} \tag{1}$$

Relation between q_1 and h_1:

$$q_1 = \frac{h_1}{R_1} \tag{2}$$

Tank-2:

Material balance:

$$q_1 - q_2 = A_2 \frac{dh_2}{dt} \tag{3}$$

Relation between q_2 and h_2:

$$q_2 = \frac{h_2}{R_2} \tag{4}$$

Writing the above equations in deviation variables $\{Q = q - q_s; Q_1 = q_1 - q_{1s}\ H_1 = h_1 - h_{1s};\ H_2 = h_2 - h_{2s}\ \}$, taking Laplace transforms, and using $A_1R_1 = \tau_1;\ A_2R_2 = \tau_2$, we get

$$\frac{H_1(s)}{Q(s)} = \frac{R_1}{\tau_1 s + 1} \tag{5}$$

$$\frac{Q_1(s)}{Q(s)} = \frac{1}{\tau_1 s + 1} \tag{6}$$

$$\frac{H_2(s)}{Q_1(s)} = \frac{R_2}{\tau_2 s + 1} \tag{7}$$

And,

$$\frac{H_2(s)}{Q(s)} = \frac{1}{\tau_1 s + 1} \frac{R_2}{\tau_2 s + 1} \tag{8}$$

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{\tau_1 s + 1} \frac{1}{\tau_2 s + 1} \tag{9}$$

Eqn.(9) can be generalized for multiple (non-interacting) systems in series as

$$\frac{Q_N(s)}{Q(s)} = \frac{K_{p1}}{\tau_1 s + 1} \frac{K_{p2}}{\tau_2 s + 1} \cdots \frac{K_{pN}}{\tau_N s + 1} \tag{10}$$

(a) Non-interacting Systems in Series (contd..)

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

For a unit-step change in Q, we obtain

$$H_2(s) = \frac{1}{s}\frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

Inversion by means of partial fraction expansion gives

$$H_2(t) = R_2\left[1 - \frac{\tau_1\tau_2}{\tau_1 - \tau_2}\left(\frac{1}{\tau_2}e^{-t/\tau_1} - \frac{1}{\tau_1}e^{-t/\tau_2}\right)\right]$$

(a) Non-interacting Systems in Series (contd..)

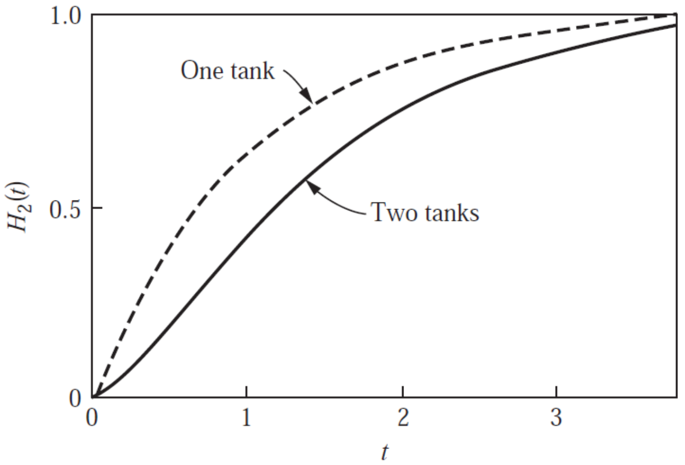

FIGURE 6–2
Transient response of liquid-level system (Example 6.1).

(a) Non-interacting Systems in Series (contd..)

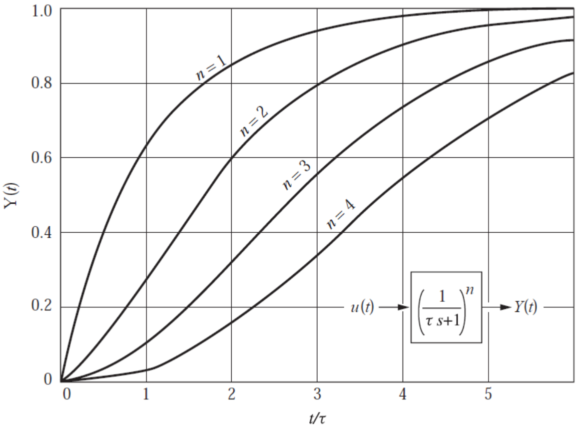

FIGURE 6–5
Step response of noninteracting first-order systems in series.

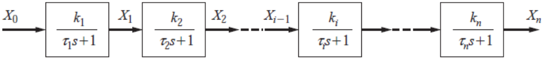

FIGURE 6–4
Noninteracting first-order systems.

(b) Interacting Systems in Series

Tank: 1
Material balance:

$$q - q_1 = A_1 \frac{dh_1}{dt} \tag{11}$$

Flow from tank-1 depends on the level difference $h_1 - h_2$. It can be written as

$$q_1 = \frac{h_1 - h_2}{R_1} \tag{12}$$

Tank: 2
Material balance:

$$q_1 - q_2 = A_2 \frac{dh_2}{dt} \tag{13}$$

Flow from tank-2 depends on the level h_2. It can be written as

$$q_2 = \frac{h_2}{R_2} \tag{14}$$

From Eqns.(11) and (12), we get

$$A_1 R_1 \frac{dh_1}{dt} + (h_1 - h_2) = R_1 q \tag{15}$$

Similarly, from Eqns.(13), (14) and (12)we get

$$A_2 R_2 \frac{dh_2}{dt} + \left(1 + \frac{R_2}{R_1}\right) h_2 - \frac{R_2}{R_1} h_1 = 0 \tag{16}$$

After rearranging the terms and taking Laplace transforms for the above equations, we get

$$(A_1 R_1 s + 1) H_1(s) - H_2(s) = R_1 Q(s) \tag{17}$$

$$-\frac{R_2}{R_1} H_1(s) + \left[A_2 R_2 s + \left(1 + \frac{R_2}{R_1}\right)\right] H_2(s) = 0 \tag{18}$$

(b) Interacting Systems in Series (contd..)

Eliminating $H_2(s)$ from Eqns.(17) and (18), and using $A_1 R_1 = \tau_1$, $A_2 R_2 = \tau_2$ we get,

$$\frac{H_1(s)}{Q(s)} = \frac{\tau_2 R_1 s + (R_1 + R_2)}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + A_1 R_2)s + 1} \tag{19}$$

Similarly, eliminating $H_1(s)$, from Eqns.(17) and (18), we get

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + A_1 R_2)s + 1} \tag{20}$$

From Eqn.(19), we could see that tank-1 is also showing second-order behavior. This is because of the effect of interaction. The response of tank-2 is second order, which is as expected. This response is slower in comparison with non-interacting system {Compare Eqns. and ... }, because of the term $A_1 R_2$ to the coefficient of s.

Comparison between Non-interacting and Interacting Systems

Non-interacting:

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)} = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1}$$

Interacting:

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + A_1 R_2)s + 1}$$

The difference between the transfer function for the non-interacting system, and that for the interacting system, is the presence of cross-product term $A_1 R_2$ in the coefficient of s.

Comparison (contd..)

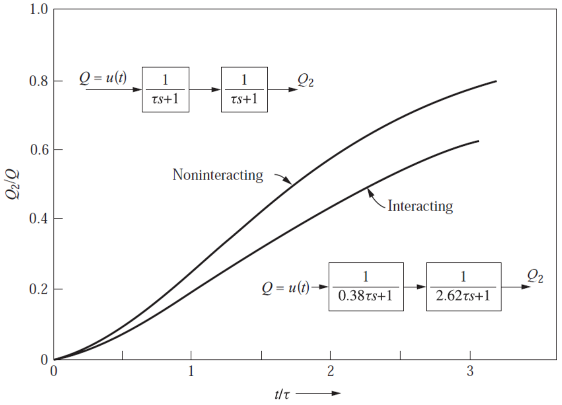

FIGURE 6–7
Effect of interaction on step response of two-tank system.

Here, $\tau_1 = \tau_2$, and $A_1 = A_2$.
Interacting system is more sluggish than the noninteracting system.