CH6605 Process Instrumentation, Dynamics and Control

First Order Systems

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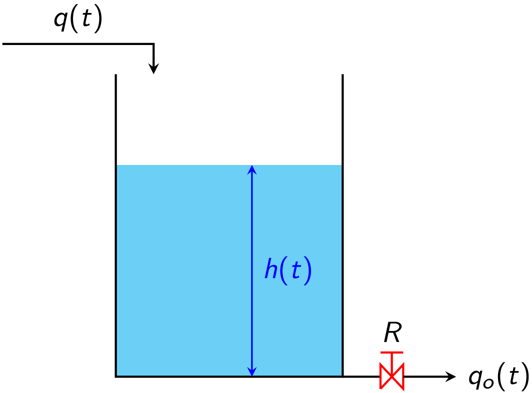
February 8, 2021

Liquid Level Variation for Changes in In-flow



Mass balance:

$$\begin{pmatrix} \text{Rate of} \\ \text{mass flow in} \end{pmatrix} - \begin{pmatrix} \text{Rate of} \\ \text{mass flow out} \end{pmatrix} = \begin{pmatrix} \text{Rate of accumulation} \\ \text{of mass in tank} \end{pmatrix}$$

$$\rho q(t) - \rho q_o(t) = \frac{d(\rho A h)}{dt}$$

Taking ρ and A as constant,

$$q(t) - q_o(t) = A\frac{dh}{dt}$$

$$q - q_o = A\frac{dh}{dt}$$

The flow rate q_o depends on the liquid level h and resistance R, as below:

$$q_o = \frac{h}{R}$$

Therefore,

$$q - \frac{h}{R} = A\frac{dh}{dt}$$

Flow through the outlet line (q_o) depends the head (h) and the total resistance (R) due to friction in the outlet pipeline and valve. It is related as

$$\Delta P = \rho g h = \frac{2fL\rho v_o^2}{D} \qquad \text{and} \quad v_o = \frac{q_o}{A_o}$$

From the above it can be noted, that

$$q_o \propto h^n$$
$$q_o = \frac{h^n}{R}$$

where R is resistance to fluid-flow.
For simplifying the derivations, we shall consider the above as

$$\boxed{q_o = \frac{h}{R}}$$

$$q - \frac{h}{R} = A\frac{dh}{dt} \tag{1}$$

Initially the process is operating at steady state, which means that $dh/dt = 0$, and with $q = q_s$, $h = h_s$. i.e.,

$$q_s - \frac{h_s}{R} = 0 \tag{2}$$

Eqn.(1) $-$ Eqn.(2) $\Longrightarrow$

$$(q - q_s) = \frac{1}{R}(h - h_s) + A\frac{d(h - h_s)}{dt} \tag{3}$$

Let us define the deviation variables as:

$$Q = q - q_s$$
$$H = h - h_s$$

Liquid Level Variation for Changes in In-flow (contd..)

Using the deviation variables in Eqn.(3), we get

$$Q = \frac{H}{R} + A\frac{dH}{dt} \tag{4}$$

Taking Laplace transform for the above, we get

$$Q(s) = \frac{H(s)}{R} + AsH(s) \tag{5}$$

Note: $\mathcal{L}[dH/dt]$ is simply $sH(s)$, because $H(0) = 0$. Rewriting the Eqn.(5) as,

$$\boxed{\frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1}} \tag{6}$$

where $\tau = AR =$ time constant of the system.

The term $\dfrac{H(s)}{Q(s)} = G(s)$ is called as the transfer function of the system.

The term R is simply the conversion factor that relates $h(t)$ to $q(t)$ when the system is at steady state. This is called as the steady state gain (K) of the system. Reason for this terminology ('steady state gain') is as follows:

Let the inflow $Q(t)$ changes according to a unit-step change (i.e., $Q(t)$ changes from its initial value of 0 to 1).

$$Q(s) = \frac{1}{s}$$

Using this in Eqn.(6), we get

$$H(s) = \frac{1}{s}\frac{R}{\tau s + 1}$$

Applying final value theorem to $H(s)$, we get

$$H(t)\Big|_{t\to\infty} = \lim_{s\to 0}[sH(s)] = \lim_{s\to 0}\frac{R}{\tau s + 1} = R$$

This shows that the ultimate change in $H(t)$ for a unit change in $Q(t)$ is simply R.

Liquid Level Variation for Changes in In-flow (contd..)

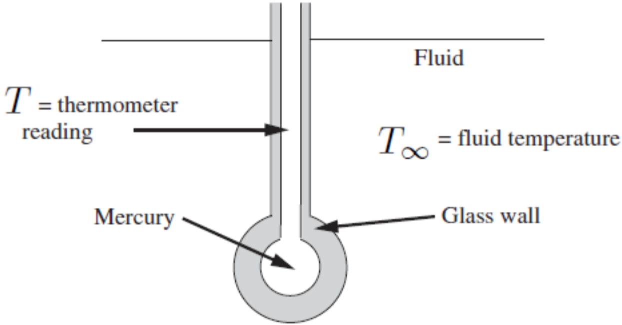
From the relation between q_o and h we can write,

$$Q_o = \frac{H}{R} \qquad \Longrightarrow \qquad \frac{Q_o(s)}{H(s)} = \frac{1}{R} \tag{7}$$

From Eqns.(6) and (7) we get,

$$\boxed{\frac{Q_o(s)}{Q(s)} = \frac{1}{\tau s + 1}}$$

Response of Thermometer



Response of Thermometer

Heat balance:

$$hA(T_\infty - T) - 0 = mC_P\frac{dT}{dt}$$

Rearranging the above,

$$\frac{mC_P}{hA}\frac{dT}{dt} + T = T_\infty$$

$$\tau\frac{dT}{dt} + T = T_\infty$$

At steady state, i.e., for $t < 0$

$$0 + T_s = T_{\infty s}$$

Using deviation variables, $\theta = T - T_s;\ \theta_\infty = T_\infty - T_{\infty s}$, we get

$$\tau\frac{d\theta}{dt} + \theta = \theta_\infty$$

Taking Laplace transform, and rearranging

$$\boxed{\frac{\theta(s)}{\theta_\infty(s)} = \frac{1}{\tau s + 1}}$$

First Order System (Generalization)

From mass or energy balances, we get for the output variable $y(t)$,

$$a_1 \frac{dy}{dt} + a_0 y(t) = bx(t)$$

$$a_1 \frac{dy}{dt} + a_0 y = bx$$

where $x(t)$ is the input. If $a_0 \neq 0$, then

$$\frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} x(t)$$

$$\tau_p \frac{dy}{dt} + y = K_p x(t) \quad (1)$$

where τ_p is known as time constant of the process, and K_p is called the steady-state gain or static gain or simply the gain of the process.

First Order System (contd..)

If $Y(t) = y(t) - y_s$ and $X(t) = x(t) - x_s$ are in terms of deviation variables around a steady state, then the initial conditions are:

$$Y(0) = y(0) - y_s = y_s - y_s = 0 \qquad \text{and} \qquad X(0) = x(0) - x_s = 0$$

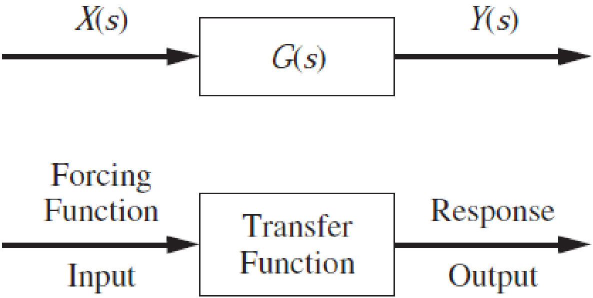
Using the above conditions, and taking the Laplace transform for Eqn.(1), we get

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

Because of the usage of deviation variables, the Laplace transform of the differential equation results in an equation that is free of initial conditions, because the initial values of X and Y are zero.

$$\frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1} = \frac{\text{Laplace transform of output deviation}}{\text{Laplace transform of output deviation}} = G(s)$$

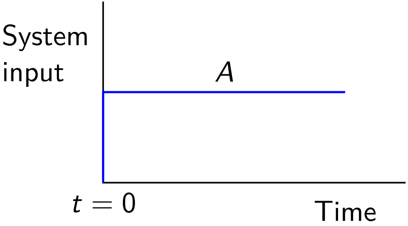
The above ratio is called the transfer function, $G(s)$, of the system. In examining physical systems, we usually attempt to obtain a transfer function.



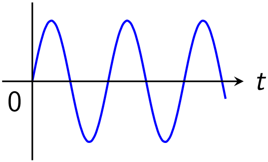
Procedure for obtaining the transfer function for a process:

- Write the appropriate balance equations (usually mass or energy balances for a chemical process).
- Linearize terms if necessary.
- Write the balance equations in deviation variable form.
- Take Laplace transform for the linear balance equations.
- Rearrange the resulting transformed equation into the transfer function form (i.e., the output divided by the input).

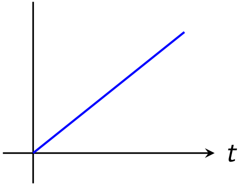
Idealized Input Functions



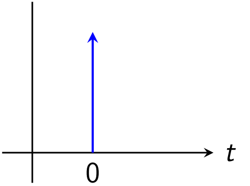
(a) step input



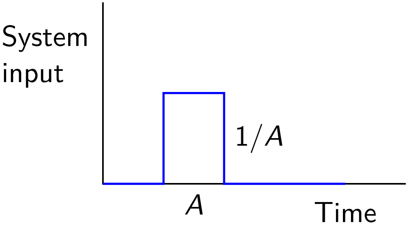
(b) sinusoidal input



(c) ramp input



(d) impulse input



(e) Unit pulse input, $\delta_A(t)$

Idealized Input Functions (contd..)

Function	$\boldsymbol{x(t)}$	$\boldsymbol{X(s)}$
Step	$x(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases}$	$\dfrac{A}{s}$
Ramp	at	$\dfrac{a}{s^2}$
Sinusoidal	$\sin(\omega t)$	$\dfrac{\omega}{s^2 + \omega^2}$
Impulse	$\delta(t)$	1
Rectangular pulse	$x(t) = \begin{cases} 0 & t < 0 \\ A & 0 < t < T \\ 0 & t > T \end{cases}$	$\dfrac{A}{s}\left(1 - e^{-sT}\right)$

First Order System - Response to Step Input

The transfer function of first order system is given by

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

Let us examine how it responds to a unit step change in input, $x(t)$.

For unit step input, $X(s) = \frac{1}{s}$. Therefore,

$$Y(s) = \frac{K_p}{s(\tau_p s + 1)} = \frac{K_p}{s} - \frac{K_p \tau_p}{\tau_p s + 1}$$

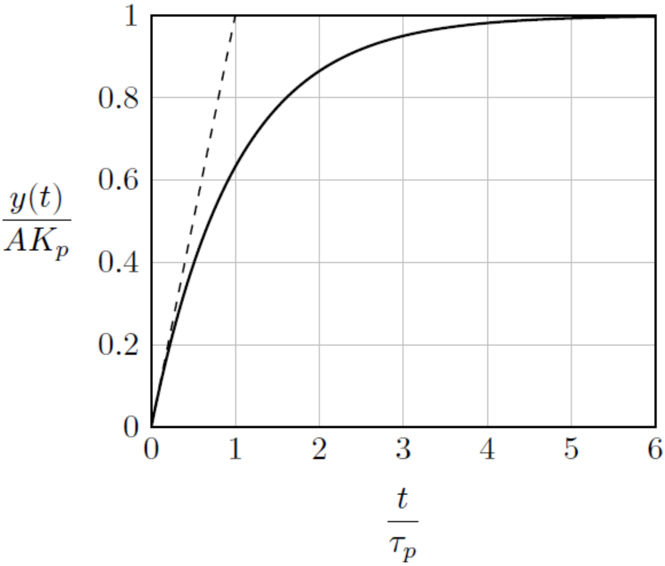
Taking inverse Laplace transform, we get

$$y(t) = K_p(1 - e^{-t/\tau_p})$$

For step input of magnitude A,

$$y(t) = AK_p(1 - e^{-t/\tau_p})$$

First Order System - Response to Step Input (contd..)



Example 1: Tank Dynamics for Step Change in In-flow

A tank of volume 0.25 m^3 and height 1 m has water flowing in at 0.05 m^3/min. The outlet flow rate is governed by the relation $F_{out} = 0.1\,h$ where h is the height of the water in the tank in m and F_{out} is the outlet flow rate in m^3/min. The inlet flow rate changes suddenly from its nominal value of 0.05 m^3/min to 0.15 m^3/min and remains there. The time (in minutes) at which the tank will begin to overflow is given by (G-2008-62)

(a) 0.28 (b) 1.01 (c) 1.73 (d) ∞

Solved Problems (contd..)

Solution:

From balance on volumetric flow rate,

$$F_i - F_o = \frac{d(Ah)}{dt} = A\frac{dh}{dt}$$

i.e.,

$$A\frac{dh}{dt} + F_o = F_i$$

Given: $F_o = F_{\text{out}} = 0.1h \text{ m}^3/\text{min}$. And, $A = V_{\text{total}}/h_{\text{total}} = 0.25/1 = 0.25 \text{ m}^2$. Therefore, the above equation becomes

$$0.25\frac{dh}{dt} + 0.1\,h = F_i \tag{1}$$

Solved Problems (contd..)

At initial steady state, $dh/dt = 0$ and, $F_i = F_o = 0.1\, h_o$.
Given: $F_i = 0.05$ m^3/min; Therefore, $h_o = 0.05/0.1 = 0.5$ m.
Rewriting the Eqn.(1) as below:

$$2.5\frac{dh}{dt} + h = 10\, F_i \tag{2}$$

At the initial steady state, the above equation becomes,

$$0 + h_o = 10 \times F_{io}$$

i.e.,

$$0 + 0.5 = 10 \times 0.05 \tag{3}$$

Eqn.(2)− Eqn.(3) $\Longrightarrow$

$$2.5\frac{dh}{dt} + (h - 0.5) = 10(F_i - 0.05)$$

Writing $(h - 0.5)$ as $\bar{h}$, and $(F_i - 0.05)$ as $\bar{F}_i$ we have

$$2.5\frac{d\bar{h}}{dt} + \bar{h} = 10\bar{F}_i \qquad \left(\text{Note: } \frac{dh}{dt} = \frac{d\bar{h}}{dt} \qquad \text{as} \quad (h - 10) = \bar{h}\right)$$

Taking Laplace transform for the above equation,

$$2.5\, s\bar{h}(s) + \bar{h}(s) = 10\bar{F}_i(s)$$

i.e.,

$$\frac{\bar{h}(s)}{\bar{F}_i(s)} = \frac{10}{2.5s + 1} = \frac{K_p}{\tau_p s + 1}$$

Here, $K_p = 10$ and $\tau_p = 2.5$.

For a step change in input of magnitude A, we get the response as

$$\bar{h}(t) = AK_p(1 - e^{-t/\tau})$$

Here, $A = 0.15 - 0.05 = 0.1\ \mathrm{m}^3/\mathrm{min}$. Therefore,

$$\bar{h}(t) = 0.1 \times 10(1 - e^{-t/2.5}) = 1 - e^{-t/2.5}$$

i.e.,

$$h - 0.5 = 1 - e^{-t/2.5}$$

The tank gets filled when h reaches the h_{total} of 1 m. i.e.,

$$1 - 0.5 = 1 - e^{-t/2.5}$$

i.e,

$$e^{-t/2.5} = 0.5$$

Taking logarithms on both sides, we get

$$-t/2.5 = \ln(0.5) \qquad \Longrightarrow \qquad t = 1.733 \text{ min}$$

The tank gets filled and starts to overflow, after 1.733 min from the start of change of flow rate to 0.15 m^3/min from its initial value of 0.05 m^3/min. **(c)** ✓

Example 2: Tank Dynamics for Step Change in In-flow (Different Solution Methods)

Consider a cylindrical tank of cross sectional area 2 m^2. Steady inflow of liquid to the tank is 0.015 m^3/s. Outflow (q_o) is related to the head (h, in m) of liquid in the tank as

$$q_o = 0.01\sqrt{h}$$

At time $t = 0$, the inflow valve is closed and so there is no inflow for $t \geq 0$. Find the time necessary to empty the tank to half the original head. Solve by: (i) direct analytical solution of differential equation, and by (ii) Laplace transform method with linearized h.

Solution:

At initial steady state, $q = q_o$. Therefore,

$$0.015 = 0.01\sqrt{h_s} \qquad \Longrightarrow \qquad h_s = 2.25 \text{ m}$$

From mass balance for the constant density systems (applicable for liquids),

$$q - q_o = A\frac{dh}{dt}$$

For $t \geq 0$, $q = 0$. Therefore,

$$-q_o = A\frac{dh}{dt}$$

Substituting for q_o and A, we get

$$-0.01\sqrt{h} = 2\frac{dh}{dt}$$

Rearranging, and integrating the above we get

$$\int_{h_s}^{h_s/2} \frac{dh}{\sqrt{h}} = -0.005 \int_0^t dt$$

$$\left[\frac{h^{[(-1/2)+1]}}{1/2}\right]_{h_s}^{h_s/2} = 0.005t$$

$$\left[2h^{1/2}\right]_{h_s}^{h_s/2} = 0.005t$$

Substituting for $h_s = 2.25$ m, we get

$$2\left[1.125^{1/2} - 2.25^{1/2}\right] = 0.005t$$

$$\implies \quad t = 175.74 \text{ s}$$

This result is obtained by direct solution of differential equation.

From mass balance for the constant density systems (applicable for liquids),

$$q - q_o = A\frac{dh}{dt}$$

At initial state,

$$0.015 - 0.01\sqrt{h_s} = 0$$

Subtracting the above two equations,

$$(q - 0.015) - (q_o - 0.01\sqrt{h_s}) = A\frac{d(h - h_s)}{dt}$$

Using deviation variables,

$$Q - Q_o = A\frac{dH}{dt} \tag{1}$$

where $Q = q - q_s = q - 0.015$, $Q_o = q_o - q_{os} = q_o - 0.01\sqrt{h_s}$ and, $H = h - h_s$.

Solved Problems (contd..)

In the above, $Q_o = q_o - 0.01\sqrt{h_s} = 0.01\sqrt{h} - 0.01\sqrt{h_s}$ i.e.,

$$Q_o = 0.01(\sqrt{h} - \sqrt{h_s}) \tag{2}$$

Since Q_o is having non-linear relation with h, i.e., $Q \propto \sqrt{h}$, we have to linearize this function before taking Laplace transform. From Taylor series expansion, for the variable $f(x)$, around x_o, and considering terms upto $f'(x_o)$, we get

$$f(x) = f(x_o) + f'(x_o)(x - x_o)$$

Here, $f(x) = \sqrt{h}$; and, $f'(x) = \dfrac{1}{2\sqrt{h}}$. Hence,

$$\sqrt{h} = \sqrt{h_o} + \frac{1}{2\sqrt{h_o}}(h - h_o)$$

Solved Problems (contd..)

For $\sqrt{h}$ around h_s, we get

$$\sqrt{h} = \sqrt{h_s} + \frac{1}{2\sqrt{h_s}}(h - h_s)$$

Using this in Eqn.(2), we get

$$Q_o = 0.01\left[\frac{1}{2\sqrt{h_s}}(h - h_s)\right]$$

We know that $h_s = 2.25$ m. Therefore,

$$Q_o = 0.01\left[\frac{1}{2\sqrt{2.25}}(h - h_s)\right] = \frac{0.01}{3}(h - h_s) = \frac{H}{300}$$

where $H = h - h_s$.

Solved Problems (contd..)

Substituting for Q_o from above in Eqn.(1), we get

$$Q - \frac{H}{300} = A\frac{dH}{dt}$$

Since, $A = 2$ m^2, we get

$$2\frac{dH}{dt} + \frac{H}{300} = Q$$
$$600\frac{dH}{dt} + H = 300Q$$

Taking Laplace transform,

$$\frac{H(s)}{Q(s)} = \frac{300}{600s+1} = \frac{K_p}{\tau_p s + 1} \qquad (2)$$

where $K_p = R = 300$ s/m^2; and, $\tau_p = AR = 2 \times 300 = 600$ s.

For $t \geq 0$, $Q(t) = q(t) - q_s = 0 - 0.015$ m^3/s. Hence, $Q(s) = -0.015/s$. Substituting this in Eqn.(2), and taking inverse Laplace transform, we get

$$H(t) = -0.015 \times 300 \times (1 - e^{-t/600})$$

Since $H(t) = h(t) - h_s = h(t) - 2.25$, we get

$$h(t) = 2.25 - 4.5 \times (1 - e^{-t/600})$$

For $h(t) = h_s/2 = 2.25/2 = 1.125$ m, we get

$$1.125 = 2.25 - 4.5 \times (1 - e^{-t/600})$$

$$\implies \quad t = 172.61 \text{ s}$$

Solved Problems (contd..)

Note: The time obtained by this Laplace transform method (i.e., 172.61 s) is slightly different from that obtained by direct solution method (i.e., 175.74 s). This is because of the approximation involved in Taylor series expansion.

The function $q = 0.01\sqrt{h}$ is approximated around its initial steady state of $h_s = 2.25$ m, as $q = \frac{0.01}{2\sqrt{h_s}}h$, i.e., $q = 0.00333h$.

Nonlinear model:

Here the outflow is given by

$$q_o = C\sqrt{h}$$

From mass balance for a constant density system,

$$q - C\sqrt{h} = A\frac{dh}{dt}$$

Linearized model:

Here the outflow is taken as

$$q_o = \frac{h}{R}$$

In terms of deviation variables, i.e., $Q = q - q_s$; and, $H = h - h_s$,

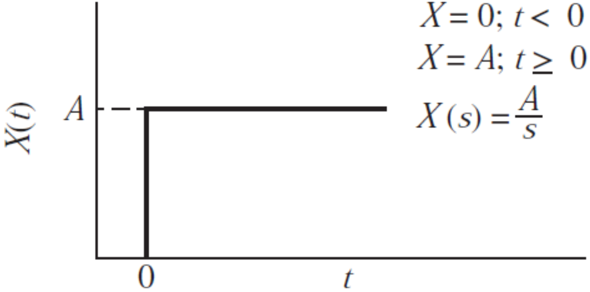
$$\frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1}$$

where

$$R = \frac{2\sqrt{h_s}}{C} \qquad \text{and} \qquad \tau = AR$$

Linearizing $\sqrt{h}$ (contd..)

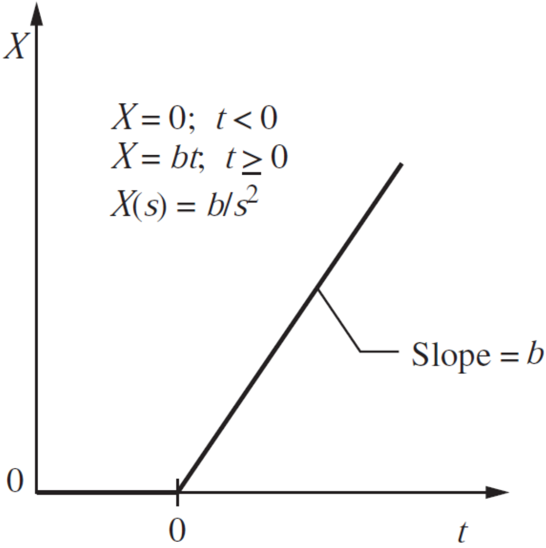
$$
\begin{aligned}
q_o &= C\sqrt{h} \\
f(h) &= C\sqrt{h} \\
f(h) &\approx f(h_s) + \left(\frac{df(h)}{dh}\right)_{h_s} (h - h_s) \\
&\approx f(h_s) + \frac{C}{2\sqrt{h_s}}(h - h_s) \\
f(h) - f(h_s) &\approx \frac{C}{2\sqrt{h_s}}(h - h_s) \\
q_o - q_{os} &= \frac{C}{2\sqrt{h_s}}(h - h_s) \\
Q_o &= \frac{C}{2\sqrt{h_s}}H
\end{aligned}
$$



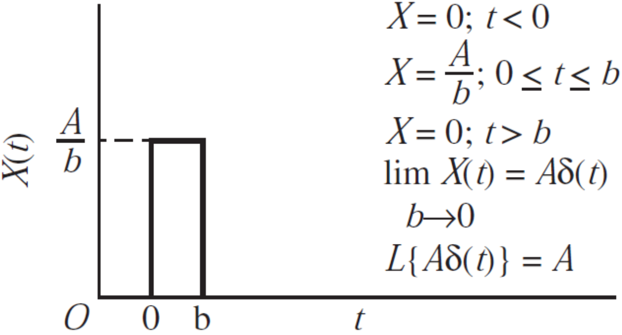
$$X = 0;\ t < 0$$
$$X = A;\ t \geq 0$$
$$X(s) = \frac{A}{s}$$

Idealized Inputs

Ramp input

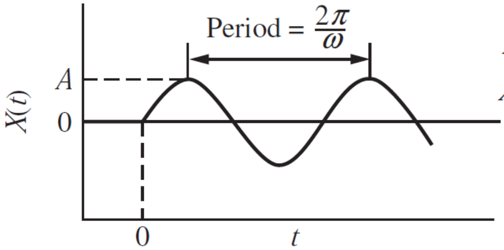


Idealized Inputs
Pulse input



Idealized Inputs

Sinusoidal input



$$X = 0;\ t < 0$$

$$X = A \sin \omega t;\ t \geq 0$$

$$X(s) = \frac{A\omega}{s^2 + \omega^2}$$

Original ODE:

$$\tau_p \frac{dy}{dt} + y = K_p x$$

ODE in deviation variable:

$$\tau_p \frac{dY}{dt} + Y = K_p X$$

where $Y = y - y_s$ is deviation variable of output; $X = x - s_s$ is deviation variable of input.
Laplace transform:

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

Step Input to First Order System

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

For step input of magnitude A, $X(t) = A$; and $X(s) = A/s$.

$$Y(s) = \frac{A}{s}\frac{K_p}{\tau_p s + 1}$$

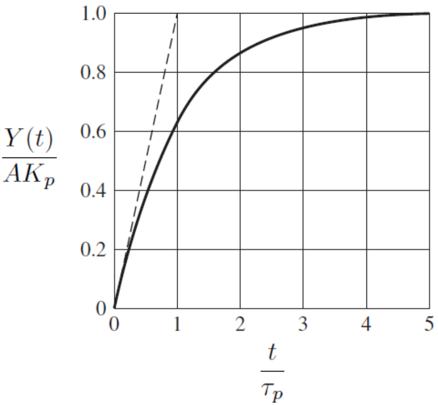
Upon partial fraction expansion, we get

$$Y(s) = AK_p \left(\frac{1}{s} - \frac{1}{s + 1/\tau_p} \right)$$

Taking $\mathcal{L}^{-1}$,

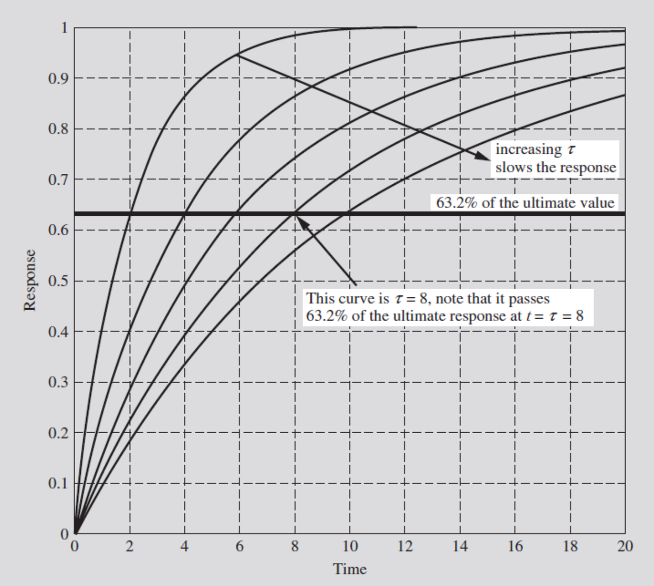
$$Y(t) = AK_p(1 - e^{-t/\tau_p})$$

Response of First Order System to Step Input



- If the initial rate of change of $Y(t)$ were maintained, the response would be complete in one time constant.
- The value of $Y(t)$ reaches 63.2% of its ultimate value when the time elapsed is equal to one time constant τ_p.

Response of First Order System to Step Input



Impulse Input to First Order System

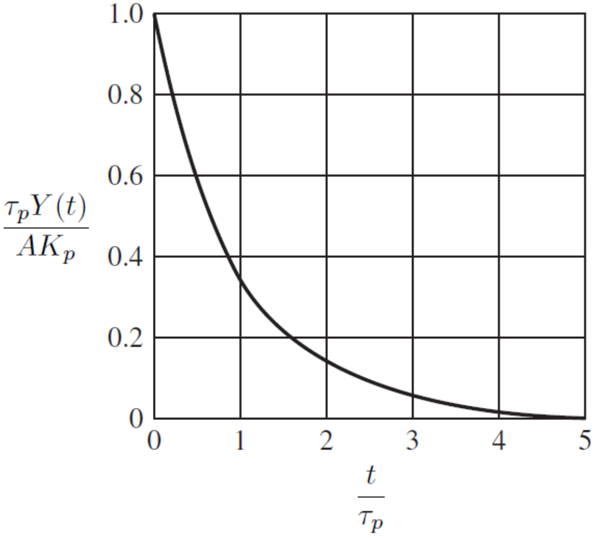
$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

For impulse input of magnitude 1, $X(s) = 1$. i.e., unit impulse.
For impulse input of magnitude A, $X(t) = A\delta(t)$, and $X(s) = A$.
This leads to

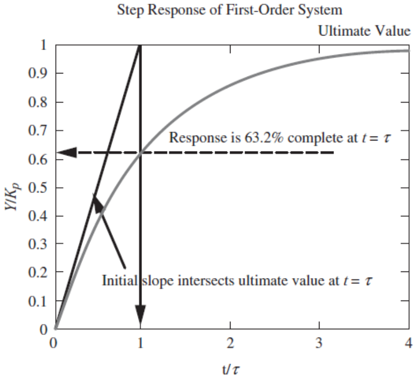
$$Y(s) = \frac{AK_p}{\tau_p s + 1} = \frac{AK_p/\tau_p}{s + 1/\tau_p}$$

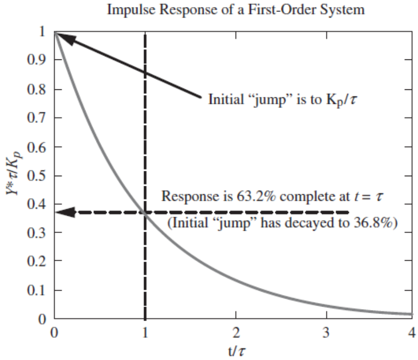
Taking $\mathcal{L}^{-1}$,

$$Y(t) = \frac{AK_p}{\tau_p}(e^{-t/\tau_p})$$


1.0
0.8
0.6
0.4
0.2
0
$\frac{\tau_p Y(t)}{AK_p}$
0
1
2
3
4
5
$\frac{t}{\tau_p}$

Comparison of Responses of Unit Step and Impulse Inputs





A certain thermocouple has a specific time constant of 2 sec. If the process temperature changes abruptly from 800 to 900°C, the temperature reading in an indicator attached to the thermocouple after 6 sec will be approximately, (G-1991-9.i)

(a) 860°C (b) 900°C (c) 890°C (d) 895°C

(d) ✓ *Explanation:* $y(t) = y_0 + A(1 - e^{-t/\tau})$. Hence, $y(t) = 800 + (900 - 800) \times (1 - e^{-t/\tau}) = 895°\text{C}$. □

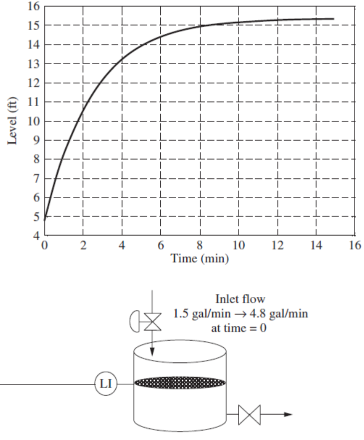
A rectangular tank is fitted with a valve at the bottom and is used for storing a liquid. The area of cross-section of the tank is 10 m^2 and the flow resistance of the valve (assumed constant) is 0.1 s/m^2. The time constant of the tank will be: (G-1988-8.c.i)

(a) 1 (b) 100 (c) 10.1 (d) 9.9

(a) ✓ *Explanation:* τ = storage capacitance $\times$ resistance to flow $= A \times R = 10 \times 0.1 = 1$ □

4.16. The level in a tank responds as a first-order system with changes in the inlet flow. Given the following level versus time data that were gathered (Fig. P4–16) after the inlet flow was

Time (min)	Level (ft)
0	4.8
0.138	5.3673
0.2761	5.9041
0.4141	6.412
0.5521	6.8927
0.6902	7.3475
0.8282	7.7779
0.9663	8.1852
1.1043	8.5706
1.2423	8.9354
1.3804	9.2805
1.5184	9.6071
1.6564	9.9161
1.7945	10.2085
1.9325	10.4853
2.0705	10.7471
2.2086	10.9949
2.3466	11.2294
2.4847	11.4513
2.6227	11.6612
2.7607	11.8599
....	....
14.3558	15.3261
14.4938	15.328
14.6319	15.3297
14.7699	15.3313



increased quickly from 1.5 to 4.8 gal/min, determine the transfer function that relates the height in the tank to the inlet flow. Be sure to use deviation variables and include units on the steady-state gain and the time constant.

4.18. Joe, the maintenance man, dumps the contents of a 55-gal drum of water into the tank process shown below.

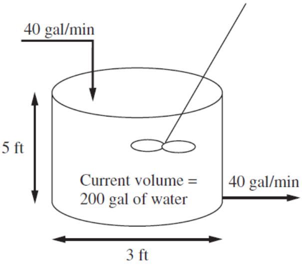


FIGURE P4–18

(a) Will the tank overflow?
(b) Plot the height as $f(t)$, starting at $t = 0$, the time of the dump.
(c) Plot the output flow as $f(t)$, starting at $t = 0$, the time of the dump.
NOTE: The output flow is proportional to the height of fluid in the tank.

Example 6: ***Dynamics of Thermometer***

A thermometer follows first-order dynamics with a time constant of 0.2 min. It is placed in a temperature bath at 100°C and is allowed to reach steady state. It is suddenly transferred to another bath at 150°C at time t = 0 and is left there for 0.2 min. It is immediately returned to the original bath at 100°C. Calculate the readings at: (i) t = 0.1 min; and, (ii) t = 0.4 min. (G-1992-19.a)

Solution:

Given: At $t = 0$, $\Delta T = 50°\text{C}$.

For first order system, fractional change in response $= 1 - e^{-t/\tau}$.

Therefore, at $t = 0.1$

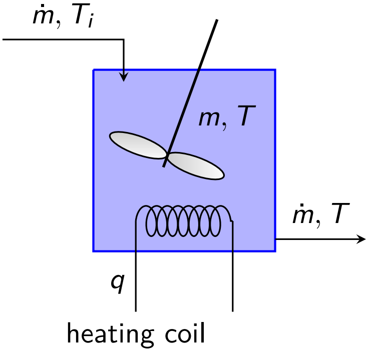
$$\begin{aligned} T &= \Delta T(1 - e^{-0.1/0.2}) + T_{\text{initial}} \\ &= 50 \times (1 - e^{-0.5}) + 100 \\ &= 119.7°\text{C} \end{aligned}$$

at $t = 0.2$

$$\begin{aligned} T &= 50 \times (1 - e^{-0.2/0.2}) + 100 \\ &= 131.6°\text{C} \end{aligned}$$

At $t = 0.2$ again a step change in T is introduced. Therefore at $t = 0.4$,

$$T = (100 - 131.6) \times (1 - e^{-0.2/0.2}) + 131.6 = 111.6°\text{C}$$



Assume:

- $\dot{m}$ = mass flow in = mass flow out = constant.
- m = mass of tank contents = constant

To find: the variation of T with t for changes in q and/or T_i.

Stirred Tank Heater (contd..)

Energy balance: Let $T_{\text{ref}} = 0$

$$\dot{m}C_P T_i - \dot{m}C_P T + q = mC_P \frac{dT}{dt} \tag{1}$$

Assume C_P is a constant over the temperature range considered. At stead state, $T_i = T_{is}$; $T = T_s$; $q = q_s$; and, $dT/dt = 0$. Using these conditions in Eqn.(1), we get

$$\dot{m}C_P(T_{is} - T_s) + q_s = 0 \tag{2}$$

Eqn.(1) $-$ Eqn.(2) $\Longrightarrow$

$$\dot{m}C_P(T_i - T_{is}) - \dot{m}C_P(T - T_s) + (q - q_s) = mC_P \frac{d(T - T_s)}{dt} \tag{3}$$

Rearranging, and using deviation variables $T_i - T_{is} = T_i'$; $T - T_s = T'$; $q - q_s = Q$, we get

$$\tau \frac{dT'}{dt} + T' = T_i' + \frac{Q}{\dot{m}C_P} \tag{4}$$

where $\tau = m/\dot{m}$

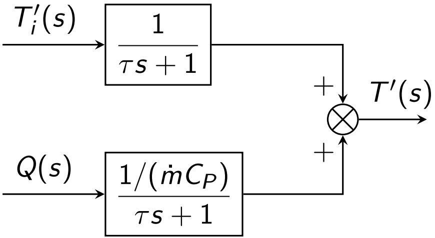
Taking Laplace transform, and rearranging,

$$T'(s) = \frac{1}{\tau s + 1} T_i'(s) + \frac{1/(\dot{m}C_P)}{\tau s + 1} Q(s)$$

Stirred Tank Heater (contd..)

$$T'(s) = \frac{1}{\tau s + 1} T_i'(s) + \frac{1/(\dot{m} C_P)}{\tau s + 1} Q(s)$$

Block Diagram Representation:



Response of First Order System to Ramp Input

First order system:

$$\frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

Input:

$$X(t) = At \qquad \Longrightarrow \qquad X(s) = \frac{A}{s^2}$$

Therefore,

$$Y(s) = \frac{A}{s^2}\frac{K_p}{\tau_p s + 1}$$

Expanding by partial fractions,

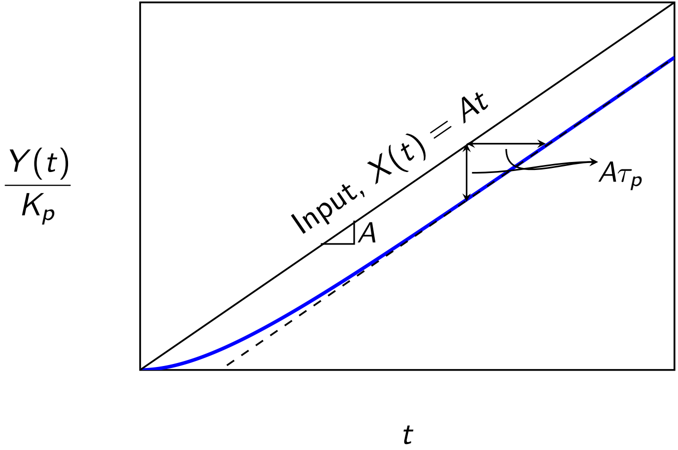
$$\frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{\tau_p s + 1} = \frac{AK_p}{s^2(\tau_p s + 1)}$$

Solving for C_1, C_2, C_3 we get

$$Y(s) = AK_p\left(\frac{1}{s^2} - \frac{\tau_p}{s} + \frac{\tau_p}{s + 1/\tau_p}\right)$$

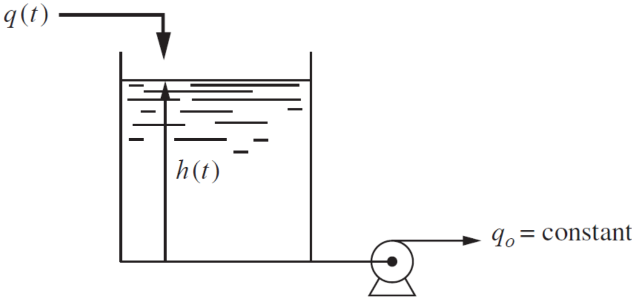
Taking inverse Laplace transform, and grouping the terms, we get

$$Y(t) = AK_p \left[t - \tau_p \left(1 - e^{-t/\tau_p} \right) \right]$$



Liquid Level System with Constant Outflow

Consider the case where there is a pump in the outflow line. Here the outflow doesn't depend on the head of liquid available, and it is constant.



From mass balance,

$$q(t) - q_o = A\frac{dh}{dt} \tag{1}$$

At steady state,

$$q_s - q_o = 0 \tag{2}$$

Subtracting Eqn.(2) from (1), and using deviation variables, we get

$$Q = A\frac{dH}{dt} \tag{3}$$

where $Q = q - q_s$; and, $H = h - h_s$. Taking Laplace transform for Eqn.(3), we get

$$\frac{H(s)}{Q(s)} = \frac{1}{As}$$

For unit step change in $Q(t)$,

$$H(t) = \frac{t}{A}$$

The step response given above is a ramp function that grows without limit.

The transfer function for the liquid-level system with constant outlet flow can be considered as a special case of first order system with $R \rightarrow \infty$.

$$\lim_{R \rightarrow \infty} \left(\frac{R}{ARs + 1} \right) = \frac{1}{As}$$