CH6605 Process Instrumentation, Dynamics and Control

Laplace Transform for Process Control

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A chemical process is a dynamical system, whose behavior changes over time. Control systems are needed to handle such changes in the process. Thus, it is important to understand the process dynamics when a control system is designed.

Mathematically, the process dynamics can be described by differential equations. Unsteady-state (or transient) process behavior corresponds to a situation, where (at least some) time derivatives of the differential equations are nonzero.

Transient operation occurs during important situations such as start-ups and shutdowns, unusual process disturbances, and planned transitions from one product grade to another.

Even at normal operation, a process does not operate at a steady state (with all time derivatives of the differential equations exactly zero) because there are always variations in external variables, such as feed composition or cooling medium temperature.

Thus, knowledge of steady-state (or static) process properties, learned in previous courses (such as thermodynamics, fluid mechanics, heat transfer, mass transfer, reaction engineering), is not sufficient for control design.

Laplace Transform

The Laplace transform of a function, $f(t)$, is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

where $F(s)$ is the symbol for the Laplace transform, $\mathcal{L}$ is the Laplace transform operator, and $f(t)$ is some function of time, t.

The $\mathcal{L}$ operator transforms a time domain function $f(t)$ into an 's' domain function, $F(s)$.

s is a complex variable: $s = x + iy$.

Laplace Transform of Common Functions

Function	$f(t)$	$F(s)$
Constant	a	$\frac{a}{s}$
Ramp	at	$\frac{a}{s^2}$
	t^n	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
	e^{at}	$\frac{1}{s-a}$
	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$

Laplace Transform of Common Functions (contd..)

Function	$f(t)$	$F(s)$
Sinusoidal	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
Sin with exponential	$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
	$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
Hyperbolic	$\sinh(\omega t)$	$\dfrac{\omega}{s^2-\omega^2}$
	$\cosh(\omega t)$	$\dfrac{s}{s^2-\omega^2}$

Laplace Transform of Common Functions (contd..)

Function	$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
Dead time	$f(t-t_o)$	$e^{-st_o}F(s)$
Impulse	$\delta(t)$	1
Rectangular pulse	$f(t)=\begin{cases} 0 & t<0 \\ A & 0<t<T \\ 0 & t>T \end{cases}$	$\dfrac{A}{s}\left(1-e^{-sT}\right)$

The *s*-differentiation rule: Multiplying $f(t)$ by t applies $-\frac{d}{ds}$ to the transform of $f(t)$.

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}\{\mathcal{L}[f(t)]\} = -\frac{d}{ds}[F(s)]$$

Example

$$\begin{aligned}
\mathcal{L}[t\sin(\omega t)] &= -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) \\
&= -\frac{(s^2+\omega^2)\times 0 - \omega\times(2s)}{(s^2+\omega^2)^2} \\
&= \frac{2\omega s}{(s^2+\omega^2)^2}
\end{aligned}$$

First shifting rule:

$$\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)]_{s \to (s-a)}$$

Example

$$\begin{aligned} \mathcal{L}[e^{at}\sin(\omega t)] &= \mathcal{L}[\sin(\omega t)]_{s \to (s-a)} \\ &= \left. \frac{\omega}{s^2 + \omega^2} \right|_{s \to (s-a)} \\ &= \frac{\omega}{(s-a)^2 + \omega^2} \end{aligned}$$

Laplace Transform of Derivatives

$$\boxed{\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)}$$

Similarly, for higher derivatives:

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

If, $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$, then

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s)$$

In process control problems, we usually assume zero initial conditions. This corresponds to the nominal steady state when "deviation variables" are used.

Laplace transform of the first two derivatives:

$$\mathcal{L}\left[\frac{dy}{dt}\right] = sY(s) - y(0)$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}\mathcal{L}[f(t)] = \frac{F(s)}{s}$$

Initial and Final Value Theorems

- **Initial Value Theorem**

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} [sF(s)]$$

It helps to determine the value of the time function $f(t)$ at $t = 0$ without finding the inverse of $F(s)$.

- **Final Value Theorem**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} [sF(s)]$$

It helps to determine the steady-state value of the system response.

Exercise Problems

1. Solve by using Laplace transform: $\dfrac{dy}{dt} = 5 - 2t; \quad y(0) = 1.$
(Ans: $y = 1 + 5t - t^2$)
2. Solve by Laplace transform method, the initial value problem $y'' = 10; \quad y(0) = y'(0) = 0.$ (Ans: $y = 5t^2$)

Not every $F(s)$ we encounter is in the Laplace table. Partial fractions is a method for re-writing $F(s)$ in a form suitable for the use of the table.

Rational Functions

A rational function is one that is the ratio of two polynomials. For example

$$\frac{s+1}{s^2+7s+9} \qquad \text{and} \qquad \frac{s^2+7s+9}{s+1}$$

are both rational functions.

A rational function is called proper if the degree of the numerator is strictly smaller than the degree of the denominator; in the examples above, the first is proper while the second is not.

The partial fraction decomposition only applies to proper functions.

Partial Fraction Expansion (contd..)

In general, if $P(s)/Q(s)$ is a proper rational function and $Q(s)$ factors into distinct linear factors
$Q(s) = (s - a_1)(s - a_2)\cdots(s - a_n)$ then

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - a_1} + \frac{A_2}{s - a_2} + \cdots + \frac{A_n}{s - a_n}$$

Use partial fractions to find $\mathcal{L}^{-1}\left(\frac{3}{s^3 - 3s^2 - s + 3}\right)$.